Finite Element Solution of Groundwater Contaminant Model

by

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OUTLINE

1. Introduce NEW Groundwater Model tracking Contaminant Dynamics in Groundwater flowing through fissures (cracks) in rock matrix via schematic diagram.

2. Present coupled PDE (Partial Differential Equation) representing model.

3. Describe terms and parameters in PDE model.

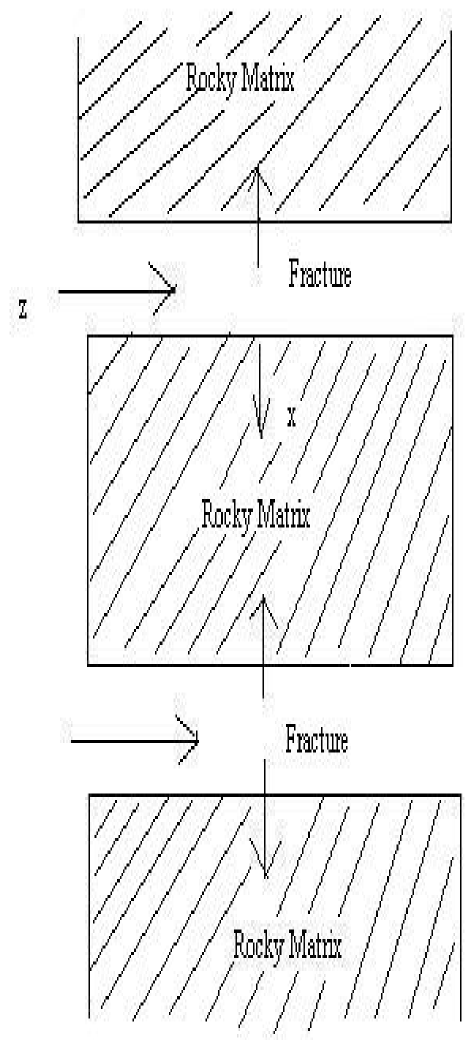
4. Describe Galerkin finite element method to numerically estimate model solution.

5. Briefly discuss convergence results.

6. Discuss use of supercomputing regarding numerical scheme.

7. Mention future Research.

SCHEMATIC DIAGRAM



PDE MODEL

$$
\begin{cases}
\partial_t C + \alpha\big(P(t)\big)\partial_z C + \beta\big(P(t)\big)\partial_{zz} C = 0 \\
\partial_t M + \gamma\big(Q(t)\big)[(\partial_{xx} + \partial_{zz})M] = 0 \\
M(t,0,z) = C(t,z); M(t,x,0) = C(t,0) = \bar{C}(t) \\
C(t,\infty) = M(t,\infty,z) = M(t,x,\infty) = 0 \\
\partial_z C(t,0) = \partial_z C(t,\infty) = 0 \\
\partial_z M(t,x,0) = \partial_z M(t,x,\infty) = 0 \\
\partial_x M(t,0,z) = \partial_x M(t,\infty,z) = 0 \\
C(0,z) = M(0,x,z) = 0
\end{cases}
$$

where

- $P(t) = \int_0^\infty C(t,z)\, dz$
- $\mathrm{Q}(t) = \int_0^\infty \int_0^\infty M(t,x,z)\, dx\, dz$
- $t \in (0,\infty)$ and $(x,y) \in (0,\infty)^2$.

FINITE ELEMENT METHOD

Define weak solution as follows:

$$\begin{cases} \langle \partial_t C,\ \phi \rangle = \alpha\big(P(t)\big)\big[\overline{C}(t) + \langle C,\ \partial_z \phi \rangle\big] + \beta\big(P(t)\big)\langle \partial_z C,\ \partial_z \phi \rangle \\ \langle \partial_t M,\ \psi \rangle = \boldsymbol{\gamma}\big(Q(t)\big)[\langle \partial_x M,\ \partial_x \psi \rangle + \langle \partial_z M,\ \partial_z \psi \rangle] \end{cases}$$

where

$$\phi \in C^1\big(t,\ z\big) \text{ and } \psi \in C^1\big(t,\ x,\ z\big)$$

together with initial condition

$$C(0, z) = M(0, x, z) = 0.$$

For computational purposes:

FINITE ELEMENT GRID

$$h_i = z_{i+1} - z_i \ \text{ for } \ i = 0, \cdots, ZDIM - 1$$

and

$$k_j = x_{j+1} - x_j \ \text{ for } \ j = 0, \cdots, XDIM - 1$$

FINITE ELEMENT APPROXIMATION

$$C(t,\ z) = \sum_{i=0}^{ZDIM} \alpha_i(t)\,\varphi_i(z)$$

and

$$M(t,\ x,\ z) = \sum_{i=0}^{ZDIM} \sum_{j=0}^{XDIM} \beta_{ij}(t)\varphi_i(z)\omega_j(x)$$

where $\{\varphi_i\}_{i=0}^{ZDIM}$ and $\{\omega_j\}_{j=0}^{XDIM}$ represent linear spline functions acting as approximating elements

COMPUTATIONAL PROBLEM

$$\begin{cases} (\dot{\vec{\alpha}}) = \mathrm{F}(\mathrm{t},\ \vec{\alpha}) \\ \vec{\alpha}(0) = \overrightarrow{(\alpha_0)} \end{cases}$$

and

$$\begin{cases} (\dot{\vec{\beta}}) = \mathrm{G}(\mathrm{t},\ \vec{\beta}) \\ \vec{\beta}(0) = \overrightarrow{(\beta_0)} \end{cases}$$

GRAPHICAL ILLUSTRATIONS

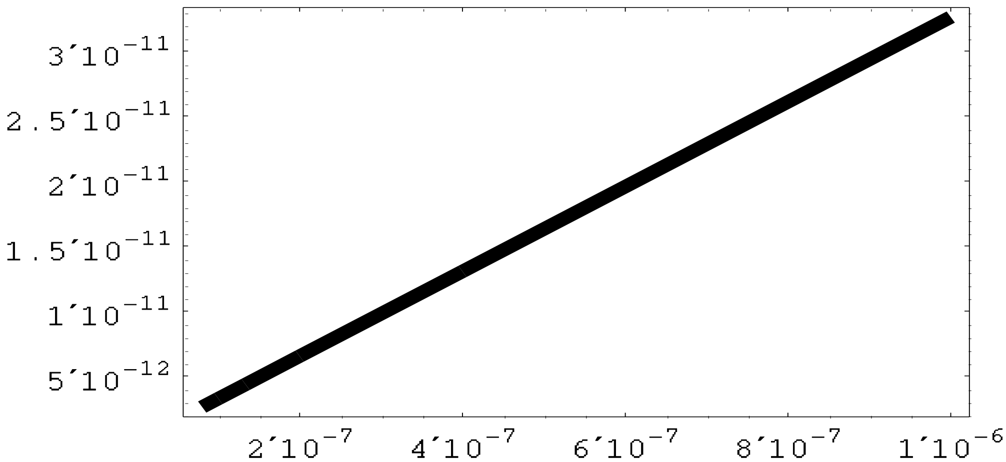


Figure 1: Square of Step Size vs Relative Error of C

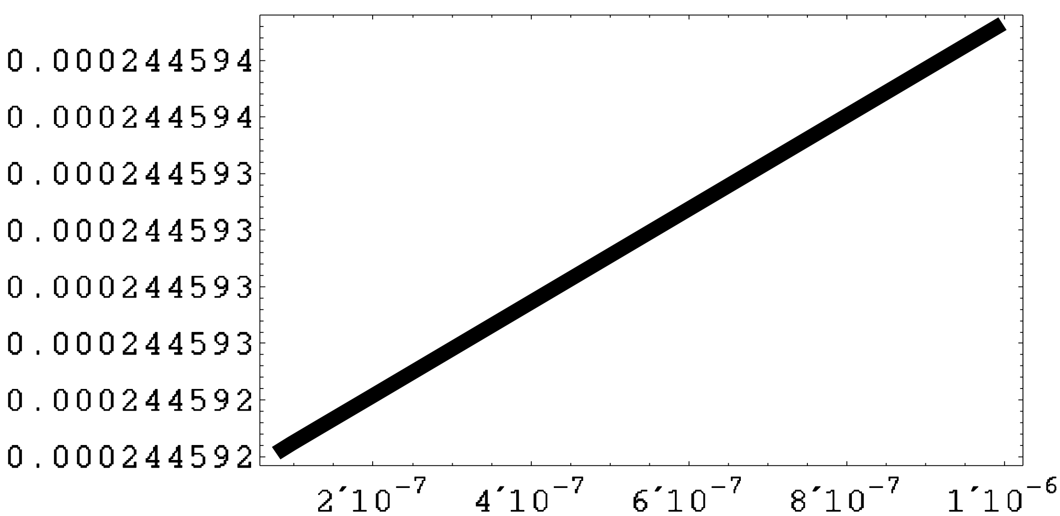


Figure 2: Square of Step Size vs Relative Error in M

FUTURE RESEARCH

1. Apply Model Results to Experimental Data
2. Introduce use of Super-Computing thereby enabling applicability to real-world much larger domains in space and time

REFERENCES

[1] E. A. Sudicky & E. O. Frind, *Contaminant Transport in Fractured Porous Media: Analytic Solutions For a System of Parallel Fractures,* Water Resources Research **18(6)**, 1982.

[2] R. L. Drake & J. Chen, *Contaminant Transport in Parallel Fractured Media: Sudicky and Frind Revisited,* Submitted for Publication, 2003.

ACKNOWLEDGMENTS

1. OSCER CONFERENCE
2. FORTRAN, C, C++ LANGUAGES
3. MATHEMATICA SOFTWARE